ON LEIBNIZ'S THEOREM ABOUT THE IMPOSSIBILITY OF SQUARING THE CIRCLE AND ITS RELATION WITH JAMES GREGORY'S *VERA CIRCULI QUADRATURA*

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1.-Introduction.

Leibniz's *De Quadratura Arithmetica Circuli, Ellipseos et Hyperbolae, cujus Corollarium est Trigonometria sine Tabulis* (hereinafter, *De Quadrature Arithmetica*)[1] is the only complete mathematical treatise written by Leibniz, and his first major achievement in infinitesimal geometry[2]. Composed between 1673 and 1676, during Leibniz's stay in Paris, the treatise was never published during Leibniz's lifetime[3].

1 I will use the letter 'A', followed by a Roman and an Arabic numerals, in order to refer to the edition of Leibniz's collected works published in the Academy Edition of Leibniz's miscellaneous works. Thus, 'AVII6' will refer to the sixth volume of the seventh tome of the Edition of the *Akademie der Wissenschaften*, and 'AVII6, 51' will refer to the text number 51 contained in that volume. In particular, AVII6, 51 contains a new critical edition of the *De Quadratura Arithmetica*, with an additional passage with respect to the first edition made by E. Knobloch in 1993 (by simplicity, I will use the shorthand 'LKQ' in order to refer to Knobloch's edition). Finally, I shall use the abbreviation 'LSG' for Gerhardt's historical edition of Leibniz's mathematical works published in seven volumes (1849-1863).
For a general overview of the text, see KNOBLOCH, 1989; for a mathematical account of its major results see, among others: KNOBLOCH, 2002.

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3 Leibniz's *De Quadratura Arithmetica* has a complex editorial history (see KNOBLOCH, 1989, PROBST, 2006), PROBST, 2008). According to Leibniz's words, the final version of the treatise was completed in the years 1675-76 (Cf. LSG, 5, 128). Before then, from the Autumn of 1673 to the Autumn of the next year, Leibniz had also composed four drafts in Latin (they are now published as: AVII4 42; AVII6 1, 3, 8 = AIII1, 39.1) and a French draft (AIII1, 39, sent to Huygens). Two drafts in French can be dated back to 1675: one was intended for La Roque (AIII1, 72) and the other supposedly for Gallois (AIII1, 73). Other drafts were composed from Spring to September 1676. These are: AVII6, 14 (fragmentary), AVII6, 20 28, and the final version, namely AVII6, 51. Conclusively, we can say that Leibniz worked on the problem of the quadrature of the circle from 1673 (AVII4, 42) to September 1676, the last month of his stay in Paris. We have to wait more than 300 years to see the first published critical edition, prepared by E. Knobloch, in 1993 (see LEIBNIZ, 1993).

One of the major results expounded in this treatise consists in the arithmetical quadratures of the circle, namely the power-series for arctan (x), together with the celebrated special cases 1/1-1/3 + 1/5-... for π/4. This result establishes the "true proportion of a circle to the circumscribed square, expressed in rational numbers", as the title of a famous paper by Leibniz recites[4].

In this article, I shall focus instead on a "negative" result proven by Leibniz in the *De Quadratura arithmetica*, namely the impossibility of finding general algebraic antiderivatives of circle-measuring integrals (theorem L, AVII6, 51, 674). I shall study the context and motivations of Leibniz's theorem, examine its content and investigate its significance with respect to Leibniz's idea of geometrical exactness.

2.- Leibniz's acquaintance with Gregory's works and his criticism.

2.1.- General background.

As it can be now ascertained, on the basis of the sixth tome of volume 7 of Leibniz's mathematical works, the Scottish mathematician James Gregory (1638-1675) exerted, through the intermediary of Christiaan Huygens (1629-1695), a tangible influence on Leibniz's impossibility theorem. In fact, in the treatise *Vera circuli et hyperbolae quadratura, in propria sua proportionis specie inventa et demonstrata* (1667), Gregory sought to prove an impossibility result close to the one we find in Leibniz's *De Quadratura Arithmetica*: the area of a sector of the circle cannot be computed "analytically" or algebraically from the radius and the chord subtending the sector (a similar result holds for the hyperbola too). In Gregory's terminology, the term "analytical" refers to any magnitude obtained by any finite composition of the five arithmetical operations, addition, subtraction, multiplication, division and root extraction to given rational magnitudes.

Gregory's starting point was a classical Archimedean procedure consisting in squeezing the circle by a double sequence of in- and circumscribed polygons. Gregory managed to symbolize this iterative polygonal construction

4 This is the "De vera proportione circuli ad quadratum circumscriptum in numeris rationalibus expressa", which appeared in the *Acta Eruditorum* in 1682. See LEIBNIZ, 2011, 7.

by means of a double converging sequence (I_n,C_n), such that each pair (I_n,C_n) could be computed algebraically from the previous pair (I_{n-1},C_{n-1}).

On this ground, he reduced a geometric problem of quadrature to an algebraic problem of computing the limit of a certain double sequence[5].

Gregory's impossibility argument thus boiled down to prove that the limit of the double sequence could not be analytically (algebraically) computed from the terms of the double sequence. From the impossibility of squaring an arbitrary sector, or impossibility of the "indefinite" quadrature of the circle, as it was called at that time, Gregory also managed to argue for the impossibility of the "definite" quadrature of the circle, namely the impossibility of finding an exact, algebraic value for the constant π[6].

In the months following the publication of the *VCHQ*, Gregory's impossibility claim was severely criticized by Christiaan Huygens. Huygens' objections to Gregory, as well as Gregory's harsh response that followed, caused a controversy between the two mathematicians which lasted for few months, from Summer 1668 to the first months of 1669. Other outstanding mathematicians and scientific personalities became involved to a different extent, such as John Wallis, Henry Oldenburg, John Collins and Lord Brouncker[7].

Huygens disagreed with Gregory's impossibility arguments on at least two grounds. First, he claimed that Gregory's inference leading to the impossibility of the analytical quadrature was valid if one assumes that the limit of the sequence of inscribed and circumscribed polygons could be computed *only* according to the special technique for the calculation of limits devised in the *VCHQ*[8]. Huygens refused to take this condition for granted, and objected that Gregory had not justified it well enough in his treatise. Without this

5 DEHN; HELLINGER, 1943, SCRIBA, 1983: 13-27, and WHITESIDE, 1961, 226-227.

6 HUYGENS, 1888-1950, vol. 6, 309; DEHN; HELLINGER, 1943, 475; LÜTZEN, 2014, 224ff.

7 The main pieces of this controversy are reproduced in HUYGENS, 1888-1950, vol. 6: n. 1605, 1647, 1648, 1653, 1669, 1682, 1684, 1685. An older edition can be found in the volume *Christiani Hugenii Zulichemii, Dum viveret Zelemii Toparchae, Opera Varia. Volumen primum. Lugduni Batavorum*, 1724. In this work, the following pieces can be found under the title *De circuli et hyperbolae quadratura Controversia : Vera Circuli et hyperbolae Quadratura authore Jacobo Gregorio* (pages 405-462); *Hugenii Observationes in librum Jacobi Gregorii, De Vera Circuli et hyperbolae quadratura* (pages 463-466); *Domini Gregorii Responsum ad animadversiones Domini Hugenii, in ejus librum, De Vera Circuli et hyperbolae quadratura* (pages 466-471); *Excerpta ex literis Domini Hugenii de responso ...* (pages 472-474); *Excerpta ex epistola D. Jacobi Gregorii, impressa in vindicationem ...* (pages 476 - 482). See also the documents and notes reproduced in TURNBULL, 1939 and, for a reconstruction of the controversy, see DIJKSTERHUIS, 1943, LÜTZEN, 2014, and CRIPPA, 2014.

8 HUYGENS, 1888-1950, vol. 6, 229. Cf. DIJKSTERHUIS, 1943, 483; LÜTZEN, 2014, 228.

justification, Gregory's impossibility proof would be incomplete. Second, Huygens objected to Gregory that he had not yet given a proof of the impossibility of squaring the *whole* circle. Huygens argued that one could not assume, as Gregory apparently did, that the impossibility of the "indefinite" quadrature of the circle entailed the impossibility of the "definite" quadrature[9]. Gregory replied to both these objections, but his answers were not considered satisfactory either by Huygens or by the other mathematicians who took part in the controversy as mediators, like Wallis. The controversy, which by the Winter of 1669 was running towards a dead-end, was then closed off almost by decree by Henry Oldenburg, the founding editor of the *Philosophical Transactions of the Royal Society* and first secretary of the Royal Society. It is worth nothing that, by that time, neither Gregory had managed to convince his adversaries, nor they had moved Gregory from his original position.

2.2.- Leibniz's acquaintance with Gregory's work.

Between 1672-1676, during his Parisian sojourn, Leibniz learned about the controversy directly from Huygens. As Huygens' notebooks record, on 30th December 1673 Leibniz borrowed a copy of the *De circuli magnitudine inventa*, Huygens work on the approximate measurement of the circumference, a copy of Gregory's *Vera Circuli et Hyperbolae Quadratura* and possibly the relevant letters concerning the controversy between Huygens and Gregory ((Huygens, 1888-1950), vol. 20, 388). Afterwards, Leibniz began to study on his own both Huygens' treatise and Gregory's *VCHQ*[10].

However, the first fair draft of the *De quadratura arithmetica*, which Leibniz sent to Huygens in Autumn 1674 with the intention of having it published in the *Journal des Sçavans*, does not bear any mention of impossibility results. On the contrary, Huygens' enthusiastic response reveals that he considered Leibniz's arithmetical quadrature as a promising step towards the discovery of the "true solution" of the problem, namely the expression of the area of the

9 HUYGENS, 1888-1950, vol. 6,. 273; DEHN; HELLINGER, 1943, 475; LÜTZEN, 2014, 230.

10 Early notes on Gregory's VCHQ can be found in AVII6, 2, 3, and in AVII5, 13. Many years later, Leibniz claimed in a letter to Wallis (28 May (or 7 June) 1697, AIII, 7, 428) that he had only skimmed through Gregory's book while in Paris. In the light of the aforementioned manuscripts this recollection appears surprising, since it appears that Leibniz studied with a certain care at least the first part of Gregory's book, the one which deals with impossibility.

circle in terms of rational or surd numbers[11].

Leibniz had initially shared Huygens' optimism, and even tried to compute the sum of the arithmetical series basing himself on his studies on numerical progressions. However, he failed to achieve any significant advance[12].

Meanwhile, Gregory's impossibility theorem was explicitly mentioned during several exchanges between Leibniz and Henry Oldenburg, the secretary of the Royal Society and one of Leibniz's main recipients in Great Britain (see Hofmann, 1975, 95). In a letter to Oldenburg from October 1674, for instance, Leibniz boasted the originality and ingenuity of his discovery about the arithmetical quadrature of the circle, remarking how no one had given before him: "a progression of rational numbers, whose sum, continued to infinity, is exactly equal to the circumference of the circle"[13]. Upon reading about Leibniz's solution to the quadrature of the circle, Oldenburg remained visibly unimpressed. Instead, he warned his correspondent about Gregory's impossibility result:

> *"And you truly say that no one has so far given a progression of rational numbers, whose sum, continued to infinity, is exactly equal to the circle (...) but I must add what I have recently received from a man expert in these matters: in fact the aforementioned Gregory is already occupied with such a matter, so that he will show in one of his writings that the exactness of the quadrature cannot be obtained"*[14].

Oldenburg's allegations seem to fall off the mark in the circumstances of his discussion. As we know, Gregory's theorem concerns the impossibility of squaring the circle analytically, i.e. by a finite algebraic expresison,

11 See also HOFMANN, 1975, 82.

12 Leibniz's attempts to compute the infinite series are now published in the tracts: AVII3, 24, AVII6, 7, 90, AVII6, 11, 111.

13 LSG, I, 53.

14 The letter dates from 8th December 1674: "Quod vero ais, neminem hactenus dedisse progressionem numerorum rationalium cujus in infinitum continuata summa sit exacte aequalis circulo (...) supra dictum nempe Gregorium in eo jam esse, ut scripto probet, exactitudinem illam obtineri non posse" (LSG, 57). As Oldenburg's remarks confirm, a reissue of the *VCHQ*, which unfortunately did not survive till us, was being prepared around 1673-1674. This occasion in particular might have inspired the reference to Gregory's impossibility claim in the excerpt above.

whereas Leibniz was praising to Oldenburg only the virtues of his solution that amounted to an infinite series[15]. We cannot exclude that at the source of Oldenburg's reservations towards the mathematical achievements of his colleague lay a misunderstanding about the meaning of exactness in mathematics. If an "exact" solution to a problem is identified with a solution that requires only algebraic curves or expressions, then Gregory's theorem implies, its correctness notwithstanding, that such an exact solution to the circle-squaring problem is wholly impossible. However, in the letter to Oldenburg, Leibniz certainly meant by an "exact solution" a solution expressed by an infinite series obeying a well-formed rule. Leibniz was thus justified in claiming that the circle-squaring problem could be exactly solved in the way he had discovered, even if he had not yet discovered whether his series for the arithmetical quadrature of the circle yielded a definite rational or irrational sum.

2.3.- Leibniz's criticism of Gregory.

We can suppose that, probably alerted and interested by Oldenburg's reply, Leibniz started a systematic discussion on the degree of exactness that different solutions to the circle-squaring problem could attain, and in a parallel way a critical study of Gregory's arguments.

A series of drafts of a letter to Oldenburg written in March 1675 informs us more precisely regarding Leibniz's general dissatisfaction with the theorems of impossibility contained in the *VCHQ*. Even if Leibniz shared Huygens' dissatisfaction with Gregory's proof of his impossibility theorem, he admitted that Huygens' criticism was not persuasive enough and had not closed the question about the analytical or algebraic unsolvability of the circle-squaring problem. As a reaction, Leibniz promised new and original objections that could persuade mathematicians to further investigate the circle-squaring problem[16].

15 See also: HOFMANN, 1975, 100.

16 "Praeter objectiones ab illustri Hugenio factas, quibus nondum est satisfactum universis, habeo et ego peculiares, unde satis judicari potest, nondum geometras ab hac inquisitione desistere debere". A subsequent letter to Oldenburg, dating from 27 August 1676, shows how Leibniz had not abandon his conviction that Gregory's proofs of impossibility was imperfect and not fully rigorous: "... Ceterum ejus demonstrationi editae de impossibilitate quadraturae absolutae circuli et hyperbolae multa haud dubie desunt" (AIII1, 89, 580). Analogous

A detailed criticism of Gregory's impossibility arguments can be indeed found in three manuscripts from 1676: *Quadraturae Circuli Arithmeticae Pars Secunda* (AVII6, n. 28, dated June or July 1676), *Series convergentes seu substitutrices* (AVII3, 60, from June 1676), and *Series convergentes duae* (AVII3, 64, June 1676). Moreover, the existence of a connection between Leibniz's own argument and his ongoing criticism of Gregory is confirmed especially by the manuscript AVII6, 28, a draft of the *De quadratura Arithmetica* from late Spring 1676. Leibniz concluded it with a *Scholium* containing a long critical discussion of the purported flaws in Gregory's impossibility argument[17].

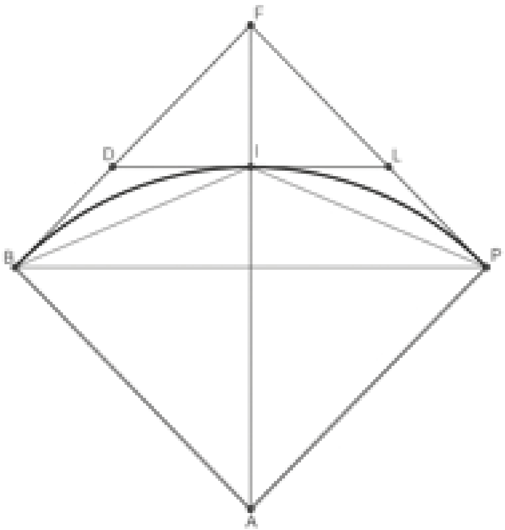

Figure 1. Gregory's construction of a series of inscribed and circumscribed polygons. Only the first and second pair are represented in figure: the inscribed triangle APB and the circumscribed quadrilateral APFB, and the inscribed quadrilateral ABIP and the circumscribed pentagon ABDLP.

Leibniz's account of Gregory's errors begins by making the point about

remarks can be found in AVII6, 19, 175: "Hanc impossibilem esse asseruit ingeniosissimus Gregorius in libro de Vera Circuli Quadratura, sed demonstrationem tunc quidem, ni fallor, non absolvit".

17 The *Scholium* does not figure in the final version of the *De Quadratura Arithmetica*. In removing the whole passage, Leibniz probably obeyed to the precise editorial policy, consisting in separating all historical digressions or philosophical notes from the mathematical content of the *De Quadratura Arithmetica*, and grouping them all together in an introduction, never finished, excerpts from which can be found in: AVII6, 39, 40, 49.

Gregory's strategy (his "*vis argumenti*") for proving the impossibility of squaring analytically a sector of a central conic. Leibniz correctly observed that the gist of Gregory's approach to the quadrature of the central conic sectors consisted in reducing the geometric problem of approximating the area of the sector by polygonal constructions to the problem of computing the limit of a certain convergent sequence (AVII6, 28, 352).

As I have recalled above, Gregory took the polygonal sequences (I_n) and (C_n), approaching a sector of the circle (a similar construction holds for a sector of a hyperbola and an ellipse) from below and from above (fig. 1). Gregory proved, in the case of a circular or elliptical sector, that: $(C_{n+1} - I_{n+1}) < 1/4\ (C_n - I_n)$, from which he concluded that these sequences converge to a limit S, namely the area of the sector (*VCHQ*, from propositions I to VI, 11-19 AVII6, 28, 352). Gregory also proved that each pair (In,Cn) is defined by the following recursive formula:

$$(1)\quad I_n = \sqrt{(I_{n-1})\,(C_{n-1})}$$
$$(2)\quad C_n = (2I_{n-1})\,(C_{n-1}) / (I_{n-1}) + \sqrt{(I_{n-1})\,(C_{n-1})}$$

Having obtained such analytical representation of the geometric polygonal construction, Gregory argued, in prop. XI of the *VCHQ* (*VCHQ*, 25) that the limit of the convergent series (I_n,C_n), which expresses the area of the sector, cannot be computed by a finite number of additions, subtractions, multiplications divisions and root extractions applied to the terms I_n and C_n.

In its general outlines, Gregory's argument aimed to prove the impossibility of finding a two-place, analytical composition *f* (namely a finite combination of additions, subtractions, multiplications divisions and root extractions) such that, applied to any pair and to the sector *S*, would yield the same quantity *K*. In symbols: $K=f(I_0; C_0)=f(I_1; C_1)=...=f(I_{n-1}; C_{n-1})=f(I_n; C_n)=f(S;S)$[18]. If such a composition could be found, Gregory argued by contradiction, *S* could also be found as the root of the (algebraic) equation:$K=f(S, S)$[19].

Gregory then argued that there exists no composition *f* which satisfies the above chain of equalities, and particularly the equation: $f(I_0 ; C_0)=f(I_1;C_1)$. Gregory's argument aims to show that, under an opportune parametrization, the members of this equation are polynomial of different powers, which remain different under any composition of finitely many algebraic opera-

18 Cf. LÜTZEN, 2014, 225-226.

19 *VCHQ*, XI, 25ff., and LÜTZEN, 2014, 226

tions.

This result, ultimately flawed, is supposed to establish the impossibility of the indefinite quadrature of the circle, since it holds for any sector *S* which can be approximated by an analytical sequence of polygons. From this would follow the impossibility of the definite quadrature, that is to say, the impossibility of expressing the constant π as a root of an algebraic equation[20].

According to Leibniz, even if Gregory had presented in the *VCHQ* an ingenious procedure for computing the limit of convergent series and thus approximating the area of a conic sector, Gregory's impossibility result was vitiated by a logical flaw ("he somehow sinned in the form of reasoning", AVII28, 358).

In Leibniz's view, Gregory had grounded his impossibility proof on the assumption that a convergent sequence tended to an analytical limit only if this limit could be found according to the special method prescribed by Gregory, or that any method capable of computing the limit would be eventually reducible to Gregory's procedure. Since this assumption is by no means evident, Gregory's proof of impossibility as presented in the *Vera quadratura* was incomplete.

As we have mentioned above, this argument is by no means new, since Huygens, but also Wallis levelled the same criticism during their discussions of Gregory's impossibility theorems[21].

Leibniz's criticism did not stop at this point, however. In fact, in the same tract AVII6, 28 and particularly in the contemporary manuscript *Series convergentes seu substitutrices* (AVII3, 60), he pushed on with and expanded his critical remarks by explaining Gregory's faulty arguments in the light of a mistaken distinction between "formula" and "quantity" (cf AVII3, 60, 758-759). As we read in AVII6, 28:

> *"It seems to me that I see what has induced into error this very intelligent man, and I have serious reasons to doubt, which would not have displeased Gregory himself if he were still alive. In fact he seems to have reasoned in this way (...) He will say that this is proven [i.e. that it is proven that the sector is not analytical with the sequence of inscribed and circumscribed polygons]*

20 VCHQ, 29.

21 Cf. HUYGENS, 1888-1950, vol. 3, 229. Wallis was especially outspoken in accusing Gregory of having committed a logical mistake: see BEELEY, 2012, vol. 3, 47.

since we have shown that an analytical formula formed by a and b, in the same way as from and cannot be given. I concede this. But if such a formula, analytically composed, is not given, then an analytical quantity expressed by this formula is not given either. It may be that the quantity is analytical and known, for instance a number; but the formula through which it is composed in the same way from the first pair and from the second pair of terms may be unknown and non-analytical"[22].

According to my reading, Leibniz was addressing a precise criticism, in the passage above, to Gregory's claim about the impossibility of the definite quadrature of the circle. Leibniz illustrated his objection with a simple numerical example: even if the number 3 is analytical both with respect to the numbers 4 and 6 and to the numbers 9 and 13, it could be obtained from the pairs (4,6) and (9,13) by means of a non-analytical (non-algebraic or "transcendental", in Leibniz's terminology) relation. One must admit, in fact, that there are examples of such non-analytical relations or functions, like logarithms, which can take analytical, i.e. algebraic values for certain algebraic arguments[23].

If we transpose this example to Gregory's result, then it appears that proving the non-existence of an analytical formula for computing the area of a sector of the circle (or of another conic) from the given polygonal series is not sufficient in order to prove that the area of a special sector, like the whole circle, is a non-analytical quantity with respect to the terms of the series[24].

We can compare Leibniz's objection with the content of Huygens' second reply to Gregory, from November 1668:

"It is still uncertain whether the circle and the square on its diameter are not

22 AVII6, 28, 354: "... nam et videre mihi videor, quod in errorem duxerit acutissimum Virum, et rationes dubitandi habeo graves, et ipsi ut arbitror Gregorio si in vivis esset, non displicituras. Itaque sic ille ratiocinatus esse videtur ... Imo vero inquiet, demonstratum est, quoniam ostendimus non posse dari formulam analyticam ex a. et b formatam, eodem modo quo ex , . Concedo. Si ergo non datur talis formula analytice composita; non datur quantitas analytica per hanc formulam significata. Potest enim fieri ut quantitas sit analytica et nota, verbi gratia numerus; formula autem secundum quam illa eodem modo componitur ex terminis duobus primis quo ex duobus secundis poterit esse ignota et non analytica".

23 The same problem is discussed in other related tracts. See, for instance: AVII6, 25, 297, AVI4, 78, 331, and the *Symbolismus memorabilis calculi algebraici et infinitesimalis*, from 1710 (LEIBNIZ, 2011, 275).

24 AVII3, 60, 759, AVII6, 28, 354.

commensurable, that is, having the proportion of a number to a number; and similarly as regards to the hyperbola and its inscribed rectilinear figure. It is sufficient to remark that his [i.e. Gregory's] proposition XI and its supplement do not prove anything when we determine by rational or surd numbers the quantities a and b in his convergent series; because at that point the termination could be also some similar number, without us being able to prove, by this Proposition, that it not the case inasmuch as we won't able to tell how the termination is composed by the first and second terms. For instance, if a is 1; and b, 2, how shall we prove by his Proposition XI that the termination is not ? Hence, in order to conclude that the proportion of the circle to the square of its diameter is not analytical, one had to prove not only that the sector of the circle is not analytic indefinitely to its inscribed figure, although this proof still keeps a certain beauty, but that this is also true for any definite case"[25].

There are clear similarities between the two passages, to the point that one may consider Leibniz's observations as an attempt to make Huygens' original objection more precise and therefore more persuasive thanks to the conceptual distinction between quantities and formulas.

Thus, aware of the distinction between analytical and non-analytical formulas and quantities which in his opinion tainted Gregory's argument, Leibniz opted for a "wholly new approach" (AVII3, 60, 758) to the proof of the impossibility of squaring a circular, elliptical and hyperbolic sector. His new strategy is simple: whereas Gregory set out to solve the problem of determining the area of a sector, Leibniz set out to solve the problem of determining the relation between the area of a sector and its tangent, and proved the non-analytic, or transcendental nature of this relation[26]. As we would say

25 "Il demeure encore incertain si le Cercle et le Quarré de son diametre ne sont pas commensurables, c'est à dire à raison de nombre à nombre; et de mesme en ce qui est d'une portion determinée de l'Hyperbole, et de sa figure rectiligne inscrite. Il suffit de remarquer que sa Proposition XI et son supplement ne prouvent rien lors qu'on determine les quantitez *a* et *b* dans sa progression convergente par des nombres rationels ou sourds; parce qu'alors la terminaison pourra aussi estre quelque nombre semblable, sans qu'on puisse demontrer le contraire par cette Proposition, d'autant qu'on ne pourra dire de quelle façon la terminaison est composée des premiers et des seconds termes. Par exemple, si *a* est 1; et *b*, 2; comment prouvera t on par sa Proposition XI que la terminaison n'est pas ? Pour conclure donc que la raison du Cercle au Quarré de son diametre nÕest pas analytique, il falloit demontrer non seulement que le Secteur de Cercle nÕest pas analytique indefinite à sa figure inscrite, quoyque cette demonstration ne laisse pas d'avoir sa beautè; mais que cela est vray aussi in omni casu definito..." (HUYGENS, 1888-1950, vol. 6, 273).

26 AVII3, 60, 758.

today, Leibniz's result amounts to prove the non-algebraic nature of certain function (namely the trigonometric functions sin or arcsin, and the logarithmic function, for what concerns the hyperbola).

In a note written between April and June 1676, titled: *Impossibilitas quadraturae circuli universalis*, Leibniz further clarified the meaning of the circle-squaring problem (and of its relative impossibility) in the following terms:

> *"The quadrature problem is twofold: there is a universal and a particular quadrature. The universal quadrature exhibits a rule with whose aid any portion of the circle can be measured, or with whose aid, from a given tangent (or sine) the arc or the angle can be found. And then there is the particular quadrature, which exhibits a certain part of the circumference (and those sectors, whose ratio with that part is known). Hence, if one exhibited the whole circle or the whole circumference, and nothing but these sectors whose ratio with the circumference is already known, one would not thereby achieve the desired universal quadrature"*[27].

By distinguishing "universal" and "particular" quadratures[28], Leibniz rendered explicit and precise the customary distinction, from the second half of XVIIth century onwards, between two sorts of problems related to the quadrature of a curve: on the one hand, the finding of the area included between the curve and two arbitrary coordinates ("indefinite quadrature"); on the other hand, the determination of the area of the whole figure (the problem of the "definite quadrature")[29]. In a slightly anachronistic terminology,

27 AVII6, n. 18, 165: "Quadratura duplex est, universalis et particularis: Universalis, quae regulam exhibet cujus ope quaelibet Circuli portio possit mensurari, seu cujus ope ex data tangente (vel sinu) possit inveniri arcus sive angulus. Particularis , quae certam circumferentiae portionem, (: et eas, quarum ad hanc portionem nota est ratio:) exhibet. Unde et si quis totum circulum totamve circumferentiam exhiberet, non vero nisi eas partes, quarum ad circumferentiam nota jam tum est ratio, is quadraturam, qualis desideratur, Universalem non dedisset".

28 The term "universal quadrature" was previously used by Mengoli to refer to Archimedes' quadrature of the parabola. Cf. *Novae quadrature arithmeticae* (1650):"Meditanti mihi persaepe Archimedis parabola Quadraturam, propterquam infinita triangula in continu? quadrupla proportione existentia certos limites quantitatis non excedunt; occurrit universalis illa Quadratura eiusdem argumenti occasione a Geometris demostrata, qua magnitudines infinita continuam quamlibet proportionem maioris inaequalitatis possidentes in praefinitas homogeneas quantitates colliguntur". On the intellectual relations between Mengoli and Leibniz, see MASSA, 2017.

29 It should be pointed out that Leibniz did not strictly adhere to his own terminology, and

we can say that, for Leibniz, the problem of "universal quadrature" of the circle is the problem of finding the general antiderivative of circle-measuring integrals, while the problem of the "particular quadrature of the circle" boils down to the problem of computing the constant π.

As the title of the piece AVII18 makes it clear, Leibniz claimed the impossibility of the universal quadrature of the central conic sections, namely the impossibility of finding an algebraic general antiderivative for circle-measuring integrals. This is equivalent to stating the non-algebraic nature of certain transcendental functions, such as tan(x) or arctan(x). On the other hand, he maintained that the question of the possibility or impossibility of the particular quadrature - that is to say, the question of whether the circle might be analytical with respect to, or even commensurable with, the square constructed on its diameter - was not a question that had yet been settled[30]. This opinion persists in the *De quadratura arithmetica*, where the closing proposition only refers to the universal, or general, quadrature of the circle and of the other central conic sections.

3.- An impossibility proof.

Aside from sparse notes from 1674 and 1675 (AVII3, 39, 589; AVII5, 26, 203), most of Leibniz's considerations on the impossibility of squaring a central conic section date back to 1676, where they appear in a number of manuscripts related to the quadrature of the circle (AVII6, 18, 166, AVII6, 19, 176, AVII6, 28, 350ff., AVII3, 60, 758ff.), and in a more complete form in proposition LI of the *De quadratura arithmetica*. We read there:

> *"It is impossible to find a better general quadrature of the circle, the ellipse or the hyperbola, or a relation between the arc and its chords, or between the number and its logarithm, which is more geometrical than our own. This pro-*

sometimes employed the term "general" as a synonym for "universal". A notable case is AVII6, 51, prop. LI, as I plan to expound below.

30 Regarding this concern, Leibniz affirmed in AVII6, 18: "Certas autem partes vel etiam totum Circulum (: sed non quamlibet ejus portionem:) analytice inveniri posse, nondum despero" ("I have not lost the hope yet that precise parts ("certas autem partes") or even the whole circle (but not any of its portions) can be found out analytically").

position stands as the crowning of our theory"[31].

As Leibniz makes it clear in the text, a "more geometrical" quadrature than the one he's given amounted to a solution obtained by intersecting curves, either algebraic or transcendental. By virtue of the equivalence between geometrical and algebraic curves in Descartes' geometry ((Bos, 2001), 336), a geometric solution obtained by algebraic curves would stand on a par with an "analytical" quadrature, which expresses the area of the circle or of the sector as an algebraic function of the sine or the tangent and of the radius. Leibniz acknowledged that solutions expressing the area of the circle (or of any circular sector) by a quantity or a sequence of quantities "whose nature and rule of continuation is known" should also be considered exact[32]. In particular, the solution presented in the *De quadratura arithmetica* belongs to this category (AVII6, 19, 175). Yet, Leibniz conceded that the arithmetical quadrature was not the most exact conceivable type of solution for the universal or particular circle-squaring problem, since one could certainly conceive (and some even tried to realize) the "analytical" or "geometric" quadrature of the circle and of all its sectors as the most exact or "perfect" quadrature, insofar as it does not make appeal to infinite expressions[33]. However the concept of such a perfect quadrature is explicitly ruled out by Leibniz as contradictory: "it is impossible - Leibniz stated in the *Praefatio* - to express the general relation between a

31 "Impossibile est meliorem invenire Quadraturam Circuli Ellipseos aut Hyperbolae generalem, sive relationem inter arcum et latera, numerumve et Logarithmum; quae magis geometrica sit, quam haec nostra est. Haec propositio velut coronis erit contemplationis hujus nostrae" AVII6, 51, 674; LQK, 134. As it has been suggested by the editors of AVII6, Leibniz employs a similar construction in a letter to Oldenburg from August 1676, in which we read: "Non credimus, meliorem circuli quadraturam linearem quam haec est unquam datum iri" (AVII6, 51, 520). For what concerns the relation between numbers and their logarithms, on the one hand, and the quadrature of conic sections, on the other hand, suffice it to say that Leibniz argued for the impossibility of finding an algebraic universal quadrature of the hyperbola on the grounds of the connection, discovered in 1647 by Grégoire of St. Vincent, between the hyperbolic areas of an equilateral hyperbola with equation: $xy=1$ and the natural logarithm function. In short, the impossibility of finding a general, algebraic relation between any hyperbolic sector and its corresponding tangent can be derived from the impossibility of expressing the logarithmic function in algebraic terms.

32 "Valor exprimi potest exacte, vel per quantitatem, vel per progressionem quantitatum cujus natura et continunandi modus cognoscitur", AVII6, 19, 174.

33 "Perfecta autem Quadratura illa erit quae simul sit Analytica et linearis, sive quae lineis aequabilibus, ad certarum dimensionum aequationes revocabilibus, construatur", AVII6, 19,175.

circular arc and its sine by an equation of a certain dimension”[34].

The earliest known proof that a perfect quadrature of the circle is impossible can be found already in the *Praefatio*. As regards its structure and content, the proof is very similar to the argument given in AVII51, which has recently been studied by Jesper Lützen (Lützen, 2014, 233). In order to integrate the account given by Lützen into the present inquiry, I shall present here the version of the impossibility proof given in the *Praefatio*, which can be considered the earliest argument elaborated by Leibniz. As in the *De quadratura arithmetica*, Leibniz reasoned by contradiction, and assumed that there exists an algebraic equation of finite degree *m*, expressing the relation between a circular arc *v* and its sine (AVII6, 19, 175). This curve, called *linea sinuum* or *curva sinuum*[35], is the curve which represents, in modern terminology, the equation *y=arcsin(v)*. If the latter equation were algebraic, then its roots could be constructed, according to the Cartesian canon for the construction of equations, by intersecting algebraic curves[36]. The easiest way to perform this construction is by intersecting the curve *y=arcsin(v)* with a straight line. As Leibniz explained, a simple way to construct the arcsin curve is via a pointwise construction, obtained by applying ordinatewise each sine to successive arc-lengths[37].

An explicit construction is given in proposition XLVIII of the *De quadratura arithmetica*, according to the following procedure.

Let the circular arc EFR be given (see fig. 2), wih radius ED and center D. Let an arc EF be taken on EFR, and let us take, or suppose given, a segment DB on DR, such that DB=arc(EF) (notice that the construction of the curve of the sines requires a procedure for rectifying any arc of the circumference). From B, let us trace a segment BC, orthogonal to AB, and equal to the sine FH of the arc EF.

34 “Sed relationem arcus ad sinum in universum aequatione certae dimensionis explicari impossibile est” AVII6, 19, 175.

35 Leibniz probably came to know this curve from Honoré Fabri’s treatise Opusculum geometricum de linea sinuum et cycloide, published in 1659. Cf. FABRI, 1659, 5, 10.

36 For an overview and discussion on the history of the Cartesian technique for the construction of equations, see: BOS, 1984. Leibniz was certainly familiar with this technique, and he had made interesting contributions himself (as in the *De constructione*, AVI 3, 45).

37 AVII6, 19, 175: “Hoc posito linea curva ejusdem gradus delineari poterit, ita ut abscissa exprimente sinus, ordinata exprimat arcus, vel contra. Hujus ergo lineae ope poterit arcus, vel angulus in data ratione secari, sive arcus, qui ad datum rationem habeat datam, inveniri sinus ...”.

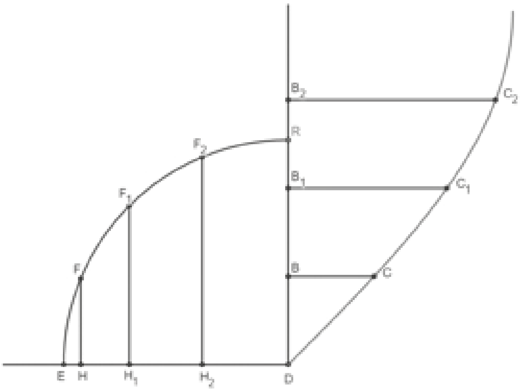

Fig. 2. Leibniz's "*Linea Sinuum*".

If we repeat the same construction for any other arc, we will determine a collection of points: C, C1, C2, each one corresponding to arcs *EF*, *EF1*, *EF2*. The *linea sinuum* will be the locus of these points[38].

Moreover, since the curve is supposed to be algebraic by virtue of the *reductio* assumption, the curve is receivable in Cartesian geometry[39].

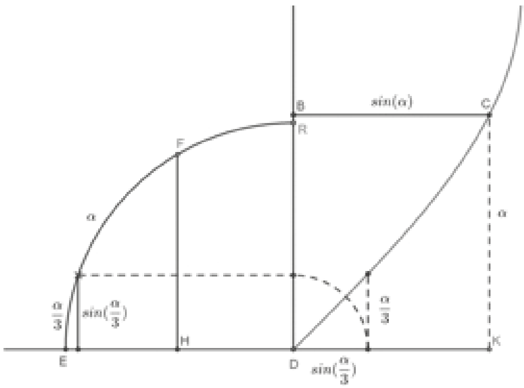

Fig.3. Trisection of the arc α (arc EF) by means of the *Linea Sinuum*.

38 AVII6, 51, 642. The procedure explained by Leibniz corresponds, in a more modern guise, to the plotting in a Cartesian reference frame, of an arbitrary number of points whose abscissas correspond to the sines of given arcs, and whose ordinates express the corresponding arc-lengths: the curve thus obtained is the arcsin function.

39 Relying on Descartes' *Géométrie*, in fact, Leibniz accepted the alleged equipollence between the expressability of a curve through an algebraic equation and its constructibility by a system of "rulers and compasses intertwined, that push and guide each other" (AIII1, 46, 204), namely articulated devices possessing one degree of freedom, so as to assure the unicity and continuity of the tracing motion. See also DESCARTES, 1897-1913, vol. 6, 391-392.

If the *linea sinuum* were a Cartesian curve, it could in principle be constructed by one and continuous motion. Thus, as was promptly noted by Leibniz, it could be successfully employed to divide any given arc not greater than a quadrant into *n* equal parts. For instance, if we want to divide the arc *EF* (fig. 3) into *n* equal parts by means of this curve, it will be sufficient to trace its sine *FH*, and construct on the extension of the radius *ED* a segment *DK=FH*. Let then the perpendicular to *DK* be constructed on point *K*, and let *C* be the intersection point between the perpendicular and the curve of the sines (supposed to be traced): the normal to *C* on *DR* will cut this axis at a point *B*. We shall then have *DB=arc*(*EF*), so that the arc *EF* is rectified. In order to solve the initial problem, it is sufficient to divide, by ruler and compass, the segment *DB* into *n* parts, and find the corresponding sines.

Since this problem has been solved by the sole used of supposedly algebraic curves, the problem of dividing an angle into an arbitrary number of parts is algebraically solvable too. But this is absurd, Leibniz insisted, because:

> *"It is well-known indeed that so many are the various degrees of the problems, as many as are the (at least odd) numbers of the sections. Indeed, bisecting an angle is a plane problem, trisecting is a solid or conic problem, dividing the angle into five parts a supersolid problem, and so further on indefinitely. The higher is the problem, the greater is the number of equal parts in which the angle must be divided. This is admitted by the Analyticians, and it could be proved universally, if we had space. Thus, it is impossible to express the relation between arc and sine universally, with a single equation of determinate degree"*[40].

In the passage above, Leibniz referred to a result on the theory of angular sections to be found in François Viète's posthumous work, *Ad angularium sectionum analyticen theoremata καθολικωτερα*[41]. Viète's treatise deals with

40 AVII6, 19, 175-176: "Constat enim tot esse varios gradus problematum, quot sunt numeri (saltem impares) sectionum; nam bisectio anguli est problema planum, trisectio problema solidum sive Conicum, quinquesectio est problema surdesolidum, et ita porro in infinitum, altius problema prout major est numerus partium aequalium, in quas dividendus est angulus; quod apud Analyticos in confesso est, et facile probari posset universaliter, si locus pateretur. Impossibile est ergo relationem arcus ad sinum, in universum certa aequatione determinati gradus exprimi".

41 The treatise was first published in 1615, with some additions by Alexander Anderson, and it was reprinted in 1646 with a slightly different title, in the edition of Viète's works edited by

what we might call, in a modern mathematical terminology, the study of the trigonometric functions of the multiples of a given circular arc or angle. The presumed aim in his booklet *Ad angularium sectionum* was to give an algebraic treatment of the relations between trigonometric lines (sines and cosines) associated with arcs and angles. As a result, he consequently tabulated the coefficients of equations expressing the relations between the sine of an angle *v* and the sine of its submultiples , for several *n* (see, for instance, Viète, 1646, 295). The schema of coefficients, constructed according to a recursive rule, was to enable him to extrapolate the equations corresponding to the division of the angle into any number *n* of parts (where *n* is an integer). In this way, Viète claimed to have given the analytical translation of the more general problem of finding: "one angle to another as one number is to another", namely the problem of the general section of the angle[42].

As Leibniz remarked, following the procedure described in the *Sectiones angulares*, each instance of the problem of dividing an arbitrary angle into *n* parts can be associated with an equation of n-th degree at most. On the basis of this result, Leibniz concluded that the problem of the general angular section could not be associated with a single polynomial equation in a finite degree. But this very problem is solved, for any *n*, by the *linea sinuum* (inasmuch as, for every *n*, this curve constructs, at most through the intersection with a straight line, the *n* -section of a given angle). Thus, it is not possible to associate this curve either with a polynomial equation in a finite, determinate degree.

There thereby arises a contradiction, from which it follows that: "it is impossible to express the relation between arc and sine universally, with a single equation of determinate degree"[43]. As a consequence, not only could Leibniz conclude the impossibility of the universal quadrature of the circle, since the relation between an arc and its corresponding sine cannot be expressed by a final algebraic equation, but was able also to establish the transcendental nature of a curve, namely the *linea sinuum*.

The proof of the same theorem given in the *De Quadratura Arithmetica* follows an analogous structure, save for the dismissal of the curve of the sines,

Frans van Schooten (cf. VIÈTE, 1646). See also: VIÈTE, 1983, 418-450.

42 VIÈTE, 1646, 300.

43 AVII6, 19, 176.

which is not strictly necessary for concluding the *reductio* argument[44], and for a reference to the division of the angle into a prime number instead of an odd number of equal parts. This small but significant change is probably a consequence of the elementary fact that the problem of dividing an angle into an odd, non-prime number *m* of equal sectors can be further reduced to the more elementary problems of dividing the angle into each prime factor of *m*.

Leibniz took for granted that equations corresponding to divisions into a prime number of parts were irreducible to lower degree equations. From our viewpoint, this claim, crucial for the completion of the impossibility argument, is by no means obvious and needs also to be adequately proved.

However, it can be supposed that early modern and later mathematicians assumed Viète's insight into the algebraic structure of the angular section problem to be correct and definitive, since we do not find any contemporary criticism on this specific point.

4.- Concluding remarks.

In the historical setting of XVIIth Century geometry, the significance of the impossibility result proved by Leibniz is at first sight not obvious, since it seems to be at odds with respect to the main activity of mathematicians at the time. This consisted, in its general outlines, in the position of problems and in their solution through a geometric construction. In asserting that there is no more geometrical quadrature than his own, in fact, Leibniz set a clear-cut limit to the type of exactness with which a solution to the universal quadrature needed to be endowed, and at the same time advised the mathematician against searching any further for a "more geometrical" solution, which would exhibit the quadrature of a sector through an equation or through a construction by geometrical curves.

Yet was a solution of this kind, expressed by an infinite series, a solution at all? It certainly was not what one might have expected as a solution to a geometric problem, because it failed to provide a construction obtained by the intersection of curves, a traditional requirement that a solution to a geometric problem would normally have been expected to fulfill[45].

44 For a reconstruction of this proof, see LÜTZEN, 2014, in particular 234-236.

45 *Scholium* XXXI, AVII6, 51, 600: "At inquies magnitudo quaesita sic non potest exhiberi, quo-

Aware of this dilemma, Leibniz noted in the same *Scholium* to proposition XXXI of the *De Quadratura Arithmetica*:

> *"I don't even promise a quadrature by means of a geometrical construction, but via an arithmetical or analytical expression. Indeed the nature of a series, even infinite, can be understood even only a few terms are understood, provided the law of formation (ratio) of the series is evident. Once this is found, it is useless to continue the series, if the point is for clarifying our understanding instead of performing a mechanical operation. If one asks for a true analytical and general relation which intervenes between the arc and the tangent, one can find in this proposition everything that can be done by Man, as I will prove below [namely, in the last proposition of the treatise]. One can find an equation of a very simple kind which expresses the dimension of the unknown quantity, whereas so far geometers have provided only approximations but not equations for the arc of the circle. I shall be silent on the fact that no one has given rational approximations to any arc or portion of the circle. Therefore, I am now the first by means of whose equation circular arcs and angles can be dealt with by an analytical calculus after the manner of straight lines"*[46].

Thus, according to Leibniz, it is sufficient to know the law of formation of an infinite series for the whole series to be exactly known. One could certainly understand a series in geometrical terms, namely as a rule for performing approximate constructions. However Leibniz also made it clear that these constructions need not be executed in order to have a better understanding of the series itself, although they can also serve for pratical purposes. Indeed, by calculating successive terms of the series (or performing the related constructions)

niam in nostra potestate non est progredi in infinitum".

46 "At inquies magnitudo quaesita sic non potest exhiberi, quoniam in nostra potestate non est progredi in infinitum. Fateor: neque enim eam constructione quadam geometrica exhibere promitto, sed expressione Arithmetica sive analytica. Seriei enim, licet infinitae, natura intelligi potest, paucis licet terminis tantum intellectis, donec progressionis ratio appareat. Qua semel inventa frustra progredimur, quoties de mente potius illustranda, quam de operatione quadam mechanica perficienda agitur. Itaque si quis veram relationem analyticam generalem quaerit quae inter arcum et tangentem intercedit, is quidem in hac propositione habet, quicquid ab homine fieri potest ut infra demonstrabo. Habet enim aequationem simplicissimi generis quae incognitae quantitatis magnitudinem exprimit cum hactenus apud geometras appropinquationes tantum, non vero aequationes pro arcu circuli demonstratae extent. Ut taceam ne appropinquationes rationales cuilibet arcui aut portioni circulari communes a quoquam fuisse datas. Quare nunc primum hujus aequationis ope arcus circulares, et anguli instar linearum rectarum analytico calculo tractari possunt." (AVII6, 51, 600).

one enters the realm of mechanical operations, useful for the practical goal of performing trigonometrical calculations without tables, and with an error as small as we please[47].

With this consideration in mind, we might relate back to the impossibility of the universal quadrature of the central conic sections the following remarks, made by Leibniz to Conring while discussing the particular quadrature of the whole circle:

> *"Perhaps my quadrature shall be published one day in France, where I left my proofs. It is not the one desired by the vulgar mathematicians, but the one they should desire. Indeed it is impossible to express by one number the ratio between the circle and the square, but an infinite series of numbers continued to the infinite is necessary, and I think a simpler series than mine cannot be given"*[48].

The impossibility of finding a perfect quadrature, that is to say the one actually desired by the "vulgar" practitioner (although even more refined mathematicians, like Huygens, Leibniz's mentor and correspondent, believed in the possibility of the perfect quadrature of the circle) establishes that the arithmetical quadrature is the solution that mathematicians *should* desire, not the one they *do* desire[49].

This is perhaps one of the major theoretical lessons that Leibniz drew from the impossibility of providing a "perfect" quadrature of the circle and the other central conic sectors. Even if these problems could not be solved by geometrical curves, it is still not impossible to obtain an "exact" solution, provided we rethink our concept of exactness in order to legitimate infinite series

47 Cf. the same *Scholium*, AVII6, 51, 600: "et si quando contemplationem ad praxin referre licebit, operationestrigonometricae, ingenti geometriae miraculo sine tabulis perfici poterunt, errore quantumlibet parvo."

48 The letter was written on 19 March 1678. Cf. AII, 1, 606: "Tetragonismus meus edetur fortasse aliquando in Gallia, ubi demonstrationes reliqui. Non est qualem desiderant Mathematici vulgo, sed qualem desiderare debent; nam rationem inter Circulum et Quadratum uno numero explicare impossibile est, opus est ergo serie numerorum in infinitum producta, nec puto simpliciorem dari posse quam mea est". It should be pointed out that no conclusions can be drawn, on the ground of the *De Quadratura Arithmetica*, concerning the quadrature of the whole circle.

49 In this sense, Leibniz anticipates a viewpoint on the role of impossibility statements that would be emphasised in XVIII century, with Montucla and Condorcet (see LÜTZEN, 2014, 244-245).

as tools in problem-solving and as proper solutions to certain problems, such as the universal quadrature of the circle and the hyperbola.

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